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Note

Existence of the Solution of a Nonlinear Integro-Differential Equation

The existence and uniqueness of the solution $u(t)$ of the equation

$$\frac{du(t)}{dt}+a(t)\,u(t)+\int_0^t ds\,k(t,s)\,u(t-s)\,u(s)=f(t), \qquad 0\leqslant t\leqslant T, u(0)=c \tag{1}$$

was studied by Chang and Day [1], and more recently by Tang and Yuan [2]. Here $a(t)$, $f(t)$, and $k(t,s)$ are known functions of t and s in $[0, T]$. Equation (1) is easily reduced to an equivalent fixed-point equation [1]:

$$\begin{aligned} u(t) &= c\,e^{-A(t)}+\int_0^t d\tau\, e^{-[A(t)-A(\tau)]} f(\tau) \\ &\quad -\int_0^t d\tau\, e^{-[A(t)-A(\tau)]}\int_0^\tau ds\,k(\tau,s)\,u(\tau-s)\,u(s) \\ &=(F(u))(t), \end{aligned} \tag{2}$$

where $A(t)=\int_0^t d\tau\, a(\tau)$. In [2], an equation in $u(t)\,e^{A(t)}$ similar to (2) was considered. However, both of the formulations are equivalent and the arguments of one are applicable to the other with obvious replacements.

With u_0 given, let $\{u_n\}$ be defined by $u_{n+1}=F(u_n)$, $n=0, 1, 2, ...$; $\{u_n\}$ will be called the iterative sequence generated by u_0. It was shown in [1] that if $a(t)\geqslant 0$, $|c|+\int_0^T dt\,|f(t)|\leqslant\frac{1}{2}$ and $\int_0^T d\tau\int_0^\tau ds\,|k(\tau,s)|<\frac{1}{2}$, then the iterative sequence generated by $u_0=F(0)$ converges uniformly to a unique solution of (1). In [2], the existence and uniqueness of the solution is established as long as $a(t)$, $f(t)$, and $k(t,s)$ are continuous functions. Existence in [2] was deduced by invoking Schauder's fixed-point theorem. The result in [1] was concluded essentially by the contraction mapping theorem. The conditions of [1, 2] describe overlapping classes of problems. For problems encountered in practice, the condition of [1] is quite restrictive while that of [2] covers a reasonably large class. However, the result of [1] is constructive and thus may be used to approximate the solution.

This note shows that the iterative sequences of the type considered in [1] converge uniformly to the unique solution of (1) with a milder assumption than that of [2]. To be precise, we assume that

(i) functions $a(t)$ and $f(t)$ are absolutely integrable on $[0, T]$; and

(ii) $\sup_{\tau\in[0,T]}\int_0^\tau ds\,|k(\tau,s)|$ exists.

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The assumed integrability of $|a(t)|$ implies that $|A(t)-A(\tau)|$ for each t, τ in $[0, T]$ is bounded by a constant independent of t and τ. Assumptions (i) and (ii) are then easily seen to imply that

$$|g(t)| = |(F(0))(t)| \leqslant \xi \tag{3}$$

and

$$\int_0^\tau ds\, |\kappa(t, \tau, s)| = \int_0^\tau ds\, |e^{-[A(t)-A(\tau)]} k(\tau, s)| \leqslant M, \tag{4}$$

where ξ and M are some constants, independent of t and τ. Let h and α be some constants such that $h \geqslant 2\xi$ and $\alpha \geqslant h^2 M/\xi$ and define the set Q as

$$Q = \left\{ v: |v(t)| \leqslant h\, e^{\alpha t} \text{ and } \int_0^T |v(t)|\, dt \text{ exists} \right\}.$$

We state the result as

THEOREM. *Let assumptions* (i) *and* (ii) *be satisfied. Then the iterative sequence generated by an arbitrary* u_0 *in* Q *converges uniformly to the unique solution* u *of* (1) *in* Q.

Proof. We divide the proof in the following four steps.

Step 1. $u_0 \in Q$ implies that $u_n \in Q$ for $n = 1, 2, 3, \ldots$.

Proof. The result will follow if $v \in Q$ implies that $F(v) \in Q$, which may be deduced by slightly adjusting the argument of Step 3, Theorem 2.1 of [2] as shown below. With $v \in Q$,

$$\begin{aligned} |(F(v))(t)| &\leqslant |g(t)| + \int_0^t d\tau \int_0^\tau ds\, |\kappa(t, \tau, s)|\, |v(\tau - s)|\, |v(s)| \\ &\leqslant \xi + h^2 M \int_0^t d\tau\, e^{\alpha\tau} \\ &\leqslant \xi + \frac{h^2 M}{\alpha} e^{\alpha t} \\ &\leqslant h\, e^{\alpha t}. \end{aligned}$$

Step 2. For each t, $|e_n(t)| = |(u_{n+1} - u_n)(t)| \leqslant (2h/n!)(2hMt)^n\, e^{\alpha t}$.

Proof. Since u_0 and u_1 are in Q, the statement is true for $n = 0$. It follows from the definitions that

$$e_{n+1}(t) = -\int_0^t d\tau \int_0^\tau ds [\kappa(t, \tau, s)\, u_{n+1}(\tau - s) + \kappa(t, \tau, \tau - s)\, u_n(\tau - s)]\, e_n(s).$$

Assuming that the estimate is valid for $e_n(t)$, and using the fact that $u_n \in Q$ for each n from Step 1, we have

$$\begin{aligned} |e_{n+1}(t)| &\leqslant \frac{2h^2}{n!}(2hM)^n \int_0^t d\tau\, \tau^n e^{\alpha\tau} \int_0^\tau ds [|\kappa(t, \tau, s)| + |\kappa(t, \tau, \tau - s)|] \\ &\leqslant \frac{2h}{n!}(2hM)^{n+1} e^{\alpha t} \int_0^t d\tau\, \tau^n \\ &= \frac{2h}{(n+1)!}(2hMt)^{n+1} e^{\alpha t}. \end{aligned}$$

The result for each n follows by induction.

Step 3. $u_n(t) \to_{n \to \infty} u(t) \in Q$, uniformly with respect to $t \in [0, T]$.

Proof. An argument is standard: From Step 2, the series $w_n(t) = \sum_{j=0}^n e_j(t)$ is absolutely and uniformly convergent for

$$\sum_{j=0}^n |e_j(t)| \leqslant 2he^{\alpha t} \sum_{j=0}^n \frac{(2hMt)^j}{j!} \xrightarrow[n \to \infty]{} 2he^{[\alpha + 2hM]t}.$$

Consequently,

$$u(t) = \lim_{n \to \infty} u_n(t) = \lim_{n \to \infty} \left[u_0 + \sum_{j=0}^{n-1} e_j(t) \right]$$

exists. Uniform convergence of $\{|w_n|\}$ implies the same for $\{w_n\}$ and hence for $\{u_n\}$. It is clear that $u \in Q$.

Step 4. u is the unique solution of (1) in Q.

Proof. From Step 3, we have

$$u(t) = \lim_{n \to \infty} u_{n+1}(t) = g(t) - \lim_{n \to \infty} \int_0^t d\tau \int_0^\tau ds\, \kappa(t, \tau, s)\, u_n(\tau - s)\, u_n(s).$$

The integrand is bounded by an integrable function $h^2 |\kappa(t, \tau, s)| e^{\alpha\tau}$. Hence, by the Lebesgue dominated convergence theorem and Step 3, we have

$$\begin{aligned} u(t) &= g(t) - \int_0^t d\tau \int_0^\tau ds\, \kappa(t, \tau, s)\, u(\tau - s)\, u(s) \\ &= (F(u))(t). \end{aligned}$$

This implies that u is a solution of (1). Let $v \in Q$ be a different solution. Then

$$\begin{aligned} \delta(t) &= u(t) - v(t) \\ &= -\int_0^t d\tau \int_0^\tau ds [\kappa(t, \tau, s)\, u(\tau - s) + \kappa(t, \tau, \tau - s)\, v(\tau - s)]\, \delta(s). \end{aligned}$$

Since u, v are in Q, $|\delta(s)| \leqslant 2he^{\alpha s}$. As in Step 2, it follows that

$$|\delta(t)| \leqslant \frac{2h}{n!} (2hMt)^n e^{\alpha t}$$

for each n, and hence $\delta(t) = 0$.

Instead of (2), one may consider the fixed-point equation,

$$u(t) = c + \int_0^t d\tau \left[f(\tau) - a(\tau)\, u(\tau) - \int_0^\tau ds\, k(\tau, s)\, u(\tau - s)\, u(s) \right]$$

which is also equivalent to (1). The arguments used in the present note lead to similar conclusions.

References

1. S. H. Chang and J. T. Day, *J. Comput. Phys.* **26**, 162 (1978).
2. T. Tang and W. Yuan, *J. Comput. Phys.* **72**, 486 (1987).

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